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When does the competitive exclusion principle hold:

A study of the system of random linear ODE in unsteady environments

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Introduction

The competitive exclusion principle in ecology was proposed by Gause in 1932, predicting that two species competing for the same niches cannot coexist at constant population values when the resources are limited. This principle is predicted by mathematical models, most of which are autonomous models including the Lotka-Volterra model under a static environment. However, observations are found in the nature that seems to contradict this model, for example, the famous "paradox of the plankton". In reality, environmental factors are not likely to be constant, so mathematicians have been trying to alter the model to imitate the real environments.
Previous mathematical and statistical literature have evaluated the validity of this principle under some special environments. Cushing (1980) investigated the periodic switching between environments using the classical Lotka-Volterra model. The author examined that two competing species could coexist in such periodic switching. This study provided scientists with some theoretical approaches to study the periodic cycle in nature, including but not limited to seasonal changing in a year. Benaim and Lobry (2016) have further considered the special case that the environment fluctuates randomly between two status which are both favorable to the same species under the Lotka-Volterra model. They have proved rigorously that, either the coexistence of the two species or even the extinction of the species favored by both environments, could occur. This result, which contradicts our common sense, violates the competitive exclusion principle. Hening and Nguyen (2019) have studied the case of a stochastic environment. The temporal variation modeled by white noise term was considered, and they proved that when either the white noise term or the growth rate resources are nonlinear, then coexistence on fewer resources than species are possible. They also generalized the case of Benaim and Lobry to when switching the environment at random times between finite possible states, and it is possible for all species to coexist.
However, no general theory of this principle has been proposed.

1 Main Objectives

Guaranteed by Hartman-Grobman Theorem, the stability of the planar autonomous system near a hyperbolic equilibrium is the same as the stability of its linearized system. In order to develop a more fundamental understanding of the random of systems of ODE, we begin with the most basic 2×2 linear system of ODE with two distinct environments:

$$\frac{d}{dt}X = \begin{cases} A_1X & t \in [2k, 2k+1], k \in \mathbb{N} \\ A_2X & t \in [2k+1, 2k+2], k \in \mathbb{N} \end{cases} \tag{1}$$

Our primary goal is to determine the stability or instability of such systems around the origin. In order to gain some feelings on various cases, we first conducted some numerical simulations on different situations, including the uniform time intervals cases as stated above, with various combinations of matrices with different essences of eigenvalues. Numerical simulations on different initial conditions were also conducted. In the next phase, we relaxed the fixed time interval settings to a random time interval settings, with the length of each flipping to be a uniformly distributed number in a certain range. We also conducted numerical simulations with high-frequency-flipping of the environments to observe if the stability of the new systems would change. In the third phase, we focused on the cases where two matrices in the systems are "close" in the sense of matrix norm, where the stability of the systems is expected to be exactly the same with both of the two matrices, no matter how demanding and weird the flipping of the two environments is: flipping according to fixed time intervals, flipping with a high frequency, or the time intervals following any probability distribution. Finally, mathematical analysis on the case where two matrices having closed norms were conducted, utilizing the method of Lyapunov Function to prove the stability of such systems.

2 Numerical Simulations Results

Results and diagrams from numerical simulations on various cases were illustrated in this section.

2.1 Fixed Time Interval

In this section, we focused on system (1).
A_1 and A_2 have two distinct negative real eigenvalues:
10 simulations with different A_1 and A_2 were conducted, and all of them yielded a stable pattern. A plausible explanation is that both systems converge to the origin in an exponential order, which could be difficulty to create an unstable pattern.
A_1 and A_2 have two complex eigenvalues with negative real parts:
20 simulations with different A_1 and A_2 were conducted, and 8/20 yielded an unstable pattern, while 12/20 were stable pattern. Since the stable spiral spin away "a little" from the origin in each period and if we limit the environment to that time interval then it is expected to be an unstable pattern.
Variation of IC:
10 simulations were conducted with fixed A_1 and A_2, while $X(0)$ was a uniformly distributed random number from the region $[-1, 1] \times [-1, 1]$. In all the situations, the stability of the solution of the system(1) remains unchanged and unstable.

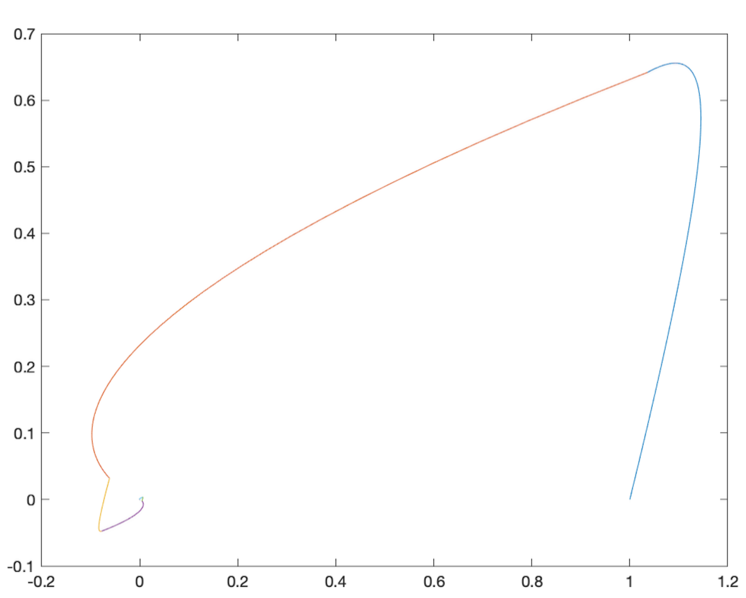

Figure 1: $A_1 = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -4 & -4 \\ 0 & -3 \end{bmatrix}$ yields a **stable** equilibrium (0,0)

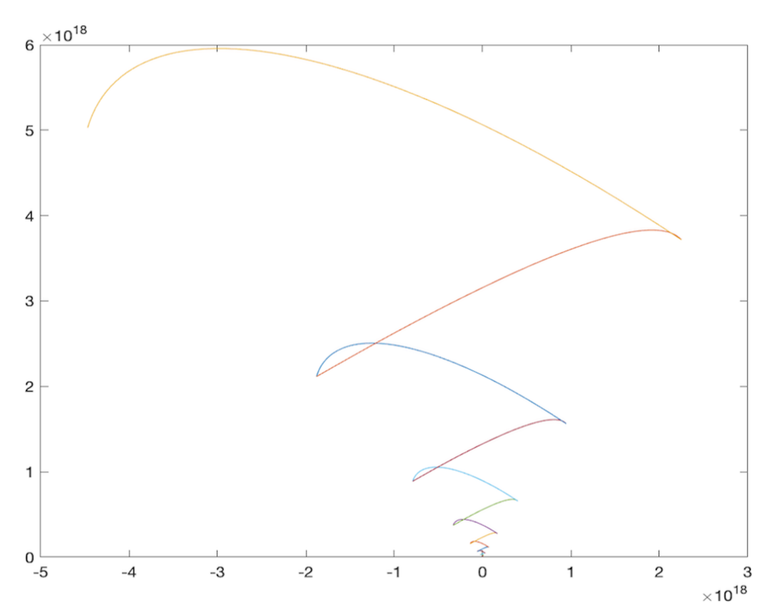

Figure 2: $A_1 = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ yields an **unstable** equilibrium (0,0)

2.2 Random Time Intervals

$$\frac{d}{dt}X = \begin{cases} A_1X & t \in I_{2k}, k \in \mathbb{N} \\ A_2X & t \in I_{2k+1}, k \in \mathbb{N} \\ I_n \sim U(0, T) & \text{for some fixed } T>0 \end{cases} \tag{2}$$

20 simulations of random time interval system (2) were conducted, among which 10 of them had a large T and 10 had small T. All the result in the large T group yielded stable pattern, and all in the small T group yielded unstable pattern.

2.3 High Frequency Flipping

Another 10 simulations were conducted using fixed and small time intervals. 7/10 yielded an unstable pattern and 3/10 yielded a stable pattern.

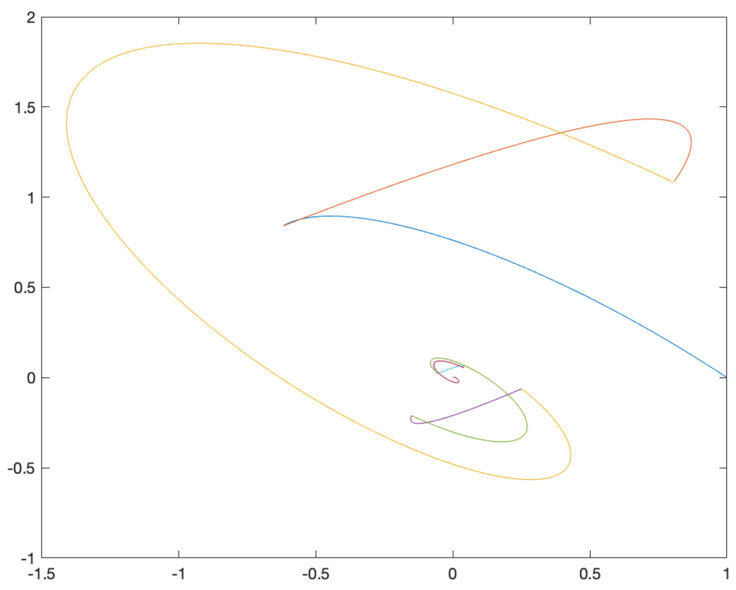

Figure 3: $A_1 = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ yields a **stable** equilibrium (0,0) when T=5

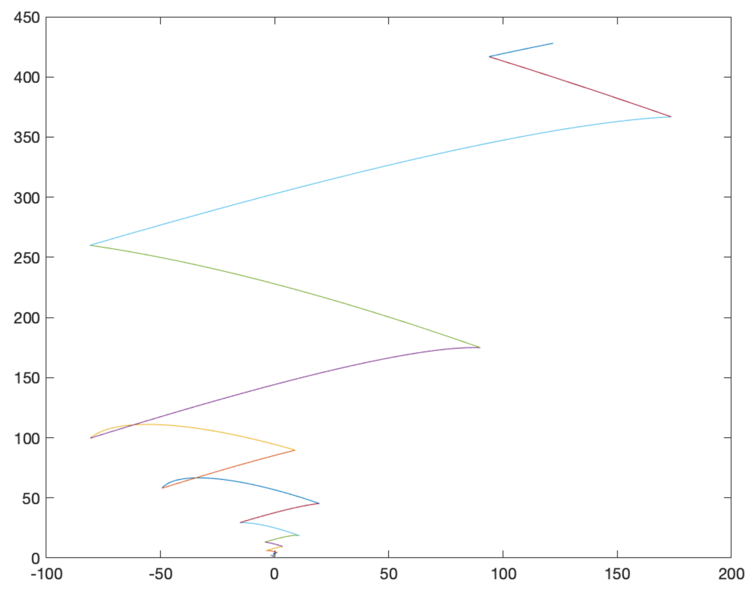

Figure 4: $A_1 = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ yields an **unstable** equilibrium (0,0) when T=1

2.4 A_1 and A_2 have close matrix norm

10 simulations were conducted, among 5 of them from system (1) and other 5 from system (2). It's not surprising that when A_1 and A_2 have close norms their phase portrait of solutions are expected to behave highly similarly. So no matter the time intervals are random or fixed, the simulations results were all expected to be stable.

3 Mathematical Analysis

3.1 Re-scaling of the colinear case

Under the system (1), when $X(0) = (x(0), y(0)) \neq (0, 0)$, $X(2) = (x(2), y(2))$ and the origin are co-linear, and $\alpha = \sqrt{x(0)^2 + y(0)^2}$, $\beta = \sqrt{x(2)^2 + y(2)^2}$, $\beta > \alpha$, then the system (1) is unstable. Moreover if we let $a_k = \sqrt{x(2k)^2 + y(2k)^2}$, where $k \in \mathbb{N}$, then $\{a_k\}$ forms a geometric series with $r = \frac{\beta}{\alpha}$.
A proof is given by change of coordinates and let $Y = \frac{\beta}{\alpha}X$.

3.2 A and A_ϵ have close matrix norm

Case 1: A **and** A_ϵ **have distinct negative real eigenvalues**
Case 2: A **and** A_ϵ **have complex eigenvalues with negative real parts**
In both cases, if $a_{ij} \neq 0, \forall i, j$, then we can find $\delta(a_{ij}) > 0$, s.t. if $||A - A_\epsilon||_{max} < \delta(a_{ij})$, (this is under the max-norm, but any norm are equivalent), then any system constructed by A and A_ϵ are stable, no matter the kind of randomness one imposes on the time intervals. The details of $\epsilon(a_{ij})$ is omitted here because it's quite tedious and could be found in the report.
A proof is given by Lyapunov's Stability Theorem and constructing a common Lyapunov Function for A and A_ϵ.

4 Conclusions and Discussions

By previous sections, we arrived at two major conclusions.
If A_1 and A_2 are two "close" enough stable systems, then no matter the way one flips the environment, the new system will always be stable. This is proved in previous section.
If A_1 and A_2 are two "distinct" enough stable systems, then if one flips the environment in a high enough frequency, one can construct an unstable system when both environmental matrices have complex eigenvalues with negative real parts. This is not proved rigorously, but illustrated by the previous numerical simulations results.
Moreover, in many unstable patterns, we observed periodic behaviors in each time interval. In the forthcoming research, focus could be put on this direction.
In general, we can also relax the constraints of 2 stable environments and further develop or expand the system to N stable environments:

$$\frac{d}{dt}X = A_iX, \text{ for } t \in I_{kN+i}, k \in \mathbb{N}, i = 1, 2, 3, ..., N \tag{3}$$

where I_n is arbitrary time interval, $n \in \mathbb{N}$

By previous result, we can always find a small enough $\delta > 0$ s.t. $||A_i - A_j||_{max} < \delta, \forall i, j \in \mathbb{N}, i, j \leq N$, then we can construct a common Lyapunov Function for all A_i satisfying the Lyapunov Stability Theorem and system(3) must be stable. In particular, if we let $N = 4$, system (3) could represent a periodic change like the four seasons in a year.
Future studies could be focused on applications of the results to a non-linear system, such as the Lotka-Volterra Model, etc.

References

1. BENAÏM, LOBRY. Lotka–Volterra with randomly fluctuating environments or "How switching between beneficial environments can make survival harder". The Annals of Applied Probability, 2016, 26.6: 3754-3785.
2. CUSHING, Jim M. Two species competition in a periodic environment. Journal of Mathematical Biology, 1980, 10.4: 385-400.
3. HENING, Alexandru; NGUYEN, Dang H. The competitive exclusion principle in stochastic environments. Journal of Mathematical Biology, 2019, 1-29.