Studying portfolio theory using MMVaR

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Abstract

We study MAXSER, a new construction of mean-variance portfolio. MAXSER formulates the mean-variance optimization problem and combines it with LASSO penalized estimation method. 4 portfolios are constructed numerically using MAXSER strategy, namely, (1) DJIA30 without factor investing, (2) DJIA30 with factor investing, (3) SP500 without factor investing and (4) SP500 with factor investing.

1 Dataset construction

1.1 Historical data

Historical data of DJIA30 constituents (30 stocks) and SP500 constituents (503 stocks) are downloaded from Yahoo Finance. Monthly excess returns are calculated based on Adjusted closing price. Historical data of DJIA index and SP index are also downloaded from Yahoo Finance. Prespecified risk constraint is calculated as the standard deviation of the monthly excess returns on DJIA30 (SP500) index during the first training period from Jan, 1995 to Dec, 1999. Historical data of Fama French 3 factors and risk-free rate from Jan, 1995 to Dec, 2019 are downloaded from Kenneth French Data Library.

1.2 Training period and testing period

Training period: T =60 months; Testing period 1: Jan, 2000 - Dec, 2019 (240 months) ; Testing period 2: Jan, 2010 - Dec, 2019(120 months) . For example, if the forecast period is Jan, 2000, the training dataset consists of monthly excess returns of stocks from Jan, 1995 to Dec, 1999.

1.3 Rolling window scheme and subpool selection

For DJIA30, at the beginning of each month, an asset pool is formed by including the constituents of the DJIA 30 index at that point in time and the FF3 factors. Portfolios are constructed using the monthly excess returns during the prior T months. A stock is excluded from the asset pool if it has missing data in the T-month training period. For SP500, at the beginning of each year, a pool of 100 stocks are randomly form from the SP500 index constituents that have complete excess monthly return data for the prior T months. We construct portfolios using the excess returns of the 100 stocks and FF3 factors over the same period. Because the number of stocks is large in this setting, when implementing MAXSER, we start with subpool selection and select a subpool containing 50 stocks. The portfolio's weights are updated for every half year (in every July, in the selected 50-stock subpool, stocks that have missing data from January to June of that year are excluded from the stock pool for the rest of the year). In both scenarios, estimate squared maximum Sharpe ratio: $\widehat{\theta}_{adj}$ for the first scenario without factor investing, $\widehat{\theta}_{all,adj}$ for the second scenario without factor investing. The subpool corresponding to the 95% quantile of estimated squared maximum Sharpe ratios is selected. (In practice, if none of 1000 subpools corresponds to the 95% quantile, among those whose quantile less than 95%, select the one with largest quantile.)

2 Practical implementation of MAXSER

2.1 Scenario 1: Without factor investing

1. **Step 1** Estimate the square of the maximum Sharpe ratio by $\widehat{\theta}_{adj}$ and compute the response $\widehat{r_c}$:

$$\widehat{\theta}_{adj} = \frac{(T-N-2)\widehat{\theta}_s - N}{T} + \frac{2\left(\widehat{\theta}_s\right)^{N/2}\left(1+\widehat{\theta}_s\right)^{-(T-2)/2}}{TB_{\widehat{\theta}_s/\left(1+\widehat{\theta}_s\right)}(N/2,(T-N)/2)}$$

where, T is the length of training period, N is the number of assets, $\widehat{\theta}_s := \widehat{\mu}'\widehat{\Sigma}^{-1}\widehat{\mu}$ is the sample estimate of θ, and

$$B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1}dy$$

$\widehat{r_c} := \sigma\frac{1+\widehat{\theta}_{adj}}{\sqrt{\widehat{\theta}_{adj}}}$

2. **Step 2** Select λ through 10-fold cross-validation. Denote the selected value by $\widehat{\lambda}$: randomly split the sample into 10 groups to form 10 validation sets. For each validation set, the training set is taken to be the rest of the observations in the sample. Next, for each such training set i, obtain the whole solution path using the method proposed in the paper *Regularization Paths for Generalized Linear Models via Coordinate Descent* by Jerome Friedman, Trevor Hastie and Rob Tibshirani from Stanford University.

LASSO implementation When implementing the lasso-penalized regression, every column of the design matrix X and the response vector y are standardized in the following manner: For each column j, our standardized variables are $Z_j = \frac{X_j - \mu_j}{s_j}$, where μ_j and s_j are the mean and standard deviation of column j, respectively. If β_j and γ_j represent the model coefficients for original data and standardized data respectively, then we should have

$$\beta_0 + \sum_{j=1}^{p}\beta_j Z_j = \gamma_0 + \sum_{j=1}^{p}\gamma_j X_j$$

$$\beta_0 + \sum_{j=1}^{p}\beta_j \frac{X_j - \mu_j}{s_j} = \gamma_0 + \sum_{j=1}^{p}\gamma_j X_j$$

$$\left(\beta_0 - \sum_{j=1}^{p}\frac{\mu_j\beta_j}{s_j}\right) + \sum_{j=1}^{p}\frac{\beta_j}{s_j}X_j = \gamma_0 + \sum_{j=1}^{p}\gamma_j X_j$$

i.e. we should have $\gamma_0 = \beta_0 - \sum_{j=1}^{p}\frac{\mu_j\beta_j}{s_j}$ and $\gamma_j = \frac{\beta_j}{s_j}$ for $j = 1,\dots,p$

Using the standardized design matrix and the response to fit the model. Re-standardize the computed weights to the original scale.

Each solution path consists of 100 λ. We choose $\lambda_{\max} = \max_\ell |\langle x_\ell, y\rangle|$, where $\langle x_\ell$ denotes the $l-th$ column of the (standardized) design matrix. Select the minimum value $\lambda_{\min} = \epsilon\lambda_{\max}$. ϵ is chosen to be 0.001. Find the value $\lambda(i)$ corresponding to the portfolio that minimizes the difference between the risk computed using the validation set and the given risk constraint. (Risk is computed as $\sqrt{w'\sigma w}$, where w is the weights computed based on training set and σ is the variance of the validation set.) The ultimate $\widehat{\lambda}$ is taken to be the average of $(\lambda(i), i = 1,\dots,10)$.

3. **Step 3** Construct the optimal LASSO-type estimator $\widehat{w^*} = \left(\widehat{w_1^*},\dots,\widehat{w_N^*}\right)'$ as follows:

$$\widehat{w^*} = \arg\min_w \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{r_c} - w'R_t\right)^2 \quad \textit{subject to} \quad \|w\|_1 \le \widehat{\lambda}$$

2.2 Scenario 2: With factor investing

1. **Step 1** Perform OLS regressions of asset returns on factor returns to obtain $\widehat{\beta}$ and $\widehat{U}$:

$$r_i = \alpha_i + \sum_{j=1}^{K}\beta_{ij}f_j + e_i =: \sum_{j=1}^{K}\beta_{ij}f_j + u_i, \quad i = 1,\dots,N$$

2. **Step 2** Compute the estimates of the square of the maximum Sharpe ratios $\widehat{\theta}_f, \widehat{\theta}_{\text{all},adj}$ and $\hat{\theta}_{u,\text{adj}}$, and compute the response $\widehat{r_c}$:

$$\widehat{\theta}_f := \widehat{\mu}_f'\widehat{\Sigma}_f^{-1}\widehat{\mu}_f \textit{ is the sample estimate of } \theta_f$$

$$\widehat{\theta}_{adj} = \frac{(T-N-K-2)\widehat{\theta}_s - N - K}{T} + \frac{2\left(\widehat{\theta}_{s,all}\right)^{(N+K)/2}\left(1+\widehat{\theta}_{s,all}\right)^{-(T-2)/2}}{TB_{\widehat{\theta}_{s,all}/\left(1+\widehat{\theta}_{s,all}\right)}((N+K)/2,(T-N-K)/2)}$$

where, T is the length of training period, N is the number of assets, K is the number of factors.

$$\widehat{\theta}_{s,all} := \widehat{\mu}'\widehat{\Sigma}^{-1}\widehat{\mu} \textit{ is the sample estimate of } \theta_s$$

where $\mu_{all} = \begin{pmatrix}\mu_f \\ \mu\end{pmatrix}$, $\quad \Sigma_{all} = \begin{pmatrix}\Sigma_f & \beta'\Sigma_f \\ \Sigma_f\beta & \Sigma\end{pmatrix}$. $\widehat{\theta}_{u,adj} := \widehat{\theta_{all,adj}} - \widehat{\theta}_f$ and $\widehat{r_c} := \frac{1+\widehat{\theta}_u}{\sqrt{\widehat{\theta}_u}}$

3. **Step 3** Like Step 2 in Scenario 1, select λ through 10-fold cross-validation. Denote the selected value by $\widehat{\lambda}$.
4. **Step 4** Like Step 3 in Scenario 1, construct the optimal LASSO-type estimator $\widehat{w_u^*}$ as follows:
$\widehat{w_u^*} = \arg\min_w \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{r_c} - w'\widehat{U}_t\right)^2$ subject to $\|w\|_1 \le \widehat{\lambda}$
5. **Step 5** Compute $\widehat{w_f^*}$ and plug in the estimates from the previous steps to obtain the MAXSER portfolio $\widehat{w_{\text{all}}}$:
$\widehat{w_f^*} := \frac{1}{\sqrt{\theta_f}}\widehat{\Sigma}_f^{-1}\widehat{\mu}_f$ and $\widehat{w_{all}} = \sigma\left(\sqrt{\frac{\widehat{\theta}_f}{\widehat{\theta}_{\text{all}}}}\widehat{w_f^*} - \sqrt{\frac{\widehat{\theta}_u}{\widehat{\theta}_{\text{all}}}}\widehat{\beta}'\widehat{w_u^*}, \sqrt{\frac{\widehat{\theta}_u}{\widehat{\theta}_{\text{all}}}}\widehat{w_u^*}\right)$

3 Numerical results

Sharpe ratios for all 4 portfolios and 2 testing periods are listed in the following table.

Portfolio	Testing period 1	Testing period 2
DJIA30 Scenario1	-0.0182	-0.0499
DJIA30 Scenario2	0.1707	0.2256
S&P500 Scenario1	-0.0432	0.2189
S&P500 Scenario2	-0.0319	0.2040

References

1. Ao, M., Yingying, L., Zheng, X. (2019). Approaching mean-variance efficiency for large portfolios. *The Review of Financial Studies, 32*(7), 2890-2919.
2. Friedman, J., Hastie, T., Tibshirani, R. (2010). Regularization paths for generalized linear models via coordinate descent. *Journal of statistical software, 33*(1), 1.