Near-miss Identities of Representations by Ternary Quadratic Forms and Spinor Class 1 Shifted Lattices

Kush Singhal

Mathematics (Intensive)

Summer Research Fellowship (SRF) 2020 for Science Students

Poster No: A6

Kush Singhal

3035552600

Mathematics (Intensive)

Abstract

This study found and classified numerous spinor class number 1 shifted lattices. Using the theory of modular forms, and a computer search, 85 shifted lattices were found whose theta series satisfied an identity indicating their spinor class number might be 1. The genera of 40 of these shifted lattices were then classified into spinor genera and classes, using the algebraic theory of quadratic lattices.

Introduction

For $\boldsymbol{a} = (a_1, a_2, a_3)^T \in \mathbb{N}^3$, the homogeneous degree 2 polynomial

$$Q_{\boldsymbol{a}}(x,y,z) = a_1x^2 + a_2y^2 + a_3z^2$$

is an example of a *ternary quadratic form*. Let

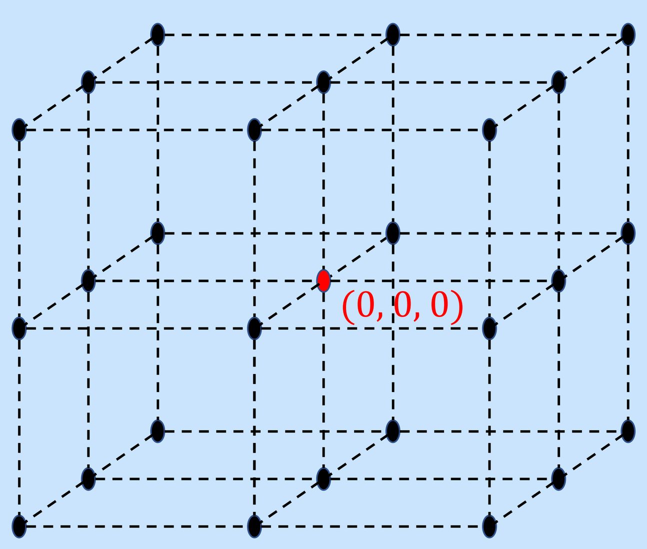
$$r_{\boldsymbol{a}}(n) := \#\{(x,y,z) \in \mathbb{Z}^3 : Q_{\boldsymbol{a}}(x,y,z) = n\}$$

$$r_{\boldsymbol{a},\boldsymbol{h},N} := \#\{\boldsymbol{x} \in \mathbb{Z}^3 : Q_{\boldsymbol{a}}(\boldsymbol{x}) = n, \boldsymbol{x} \equiv \boldsymbol{h} \pmod{N}\}$$

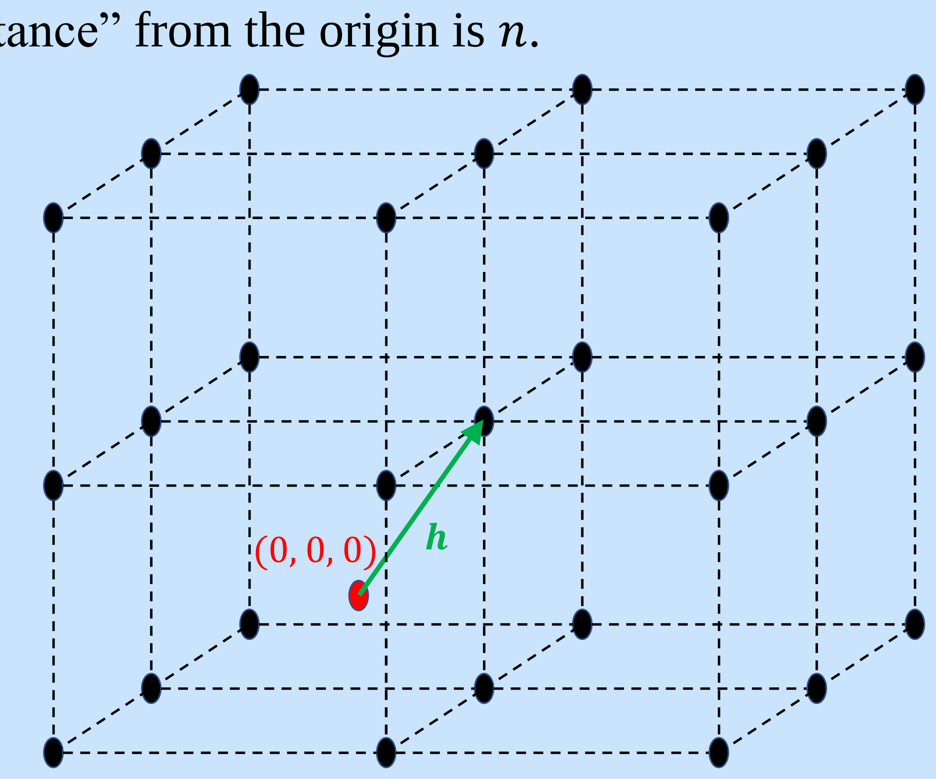
There are two approaches to studying these two quantities.

Algebraic Approach (Lattices)

- Let us think "geometrically"—let $Q_{\boldsymbol{a}}$ define some sort of (squared) 'distance function' in 3-dimensional space $\mathbb{Q}^3$; this makes $\mathbb{Q}^3$ a *ternary quadratic space* [1].
- To study $Q_{\boldsymbol{a}}(x,y,z)$ where $x,y,z \in \mathbb{Z}$, we can consider the *lattice* L of points in the quadratic space that have integer coordinates [1]. Then, $r_{\boldsymbol{a}}(n)$ is the number of lattice points whose (squared) "distance" from the origin is n.



A lattice L



A shifted lattice NL + h

- To study $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$, we may consider the *shifted lattice* (or *lattice coset*) [2]

$$NL + \boldsymbol{h} = \{N\boldsymbol{x} + \boldsymbol{h} : \boldsymbol{x} \in L\}$$

- Shifted lattices may be classified as follows [1,2]:

Genus

Two shifted lattices are in the same genus if they are essentially the same modulo power of primes, for any prime.

Spinor Genus

Class

Two shifted lattices are in the same class if one can be transformed into the other by some "distance"-preserving rotation.

Analytic Approach (Modular Forms)

- For a quadratic form $\boldsymbol{a}$, and congruence conditions $\boldsymbol{h}$ and N, we may consider the *generating functions* given by

$$\Theta_{\boldsymbol{a}}(z) = \sum_{n\geq 0} r_{\boldsymbol{a}}(n)\, q^n \quad \text{and} \quad \Theta_{\boldsymbol{a},\boldsymbol{h},N}(z) = \sum_{n \geq 0} r_{\boldsymbol{a},\boldsymbol{h},N}(n)\, q^n$$

where $q := e^{2\pi i z}$. These are examples of *theta series*.

- These theta series are *modular forms* of weight $\frac{3}{2}$ [3].
- It is conjectured in [4], that for a shifted lattice $NL + \boldsymbol{h}$, we have

$$(\text{avg. of theta series of classes in spinor genus of } NL + \boldsymbol{h}) = \mathcal{E}_{gen} + U_{spn}$$

where

  - $\mathcal{E}_{gen}$ = average of theta series over genus of $NL + \boldsymbol{h}$
  - U_{spn} = linear combination of *unary theta functions* (these are special kind of series whose only non-zero terms occur for certain special square classes)

- Thus, if $NL + \boldsymbol{h}$ has spinor class number 1 (i.e. only 1 class in the spinor genus), the conjecture boils down to

$$\Theta_{\boldsymbol{a},\boldsymbol{h},N} = \mathcal{E}_{gen} + U_{spn}$$

- This identity then gives a relationship between $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ and $r_{\boldsymbol{a}}(n)$ for all n except those n lying in certain `bad' square classes.

Goal

I. Using a computer, find values for $\boldsymbol{a}, \boldsymbol{h}, n$ for which $\Theta_{\boldsymbol{a},\boldsymbol{h},N}$ satisfies an identity of the type $\Theta_{\boldsymbol{a},\boldsymbol{h},N} = \mathcal{E}_{gen} + U_{spn}$.

II. After this, for each $\boldsymbol{a}, \boldsymbol{h}, N$ found above, try and classify the genus of the corresponding shifted lattice into spinor genera and classes.

Methods

For Goal I: Computer Search

- In this search, the computer only looked for $\boldsymbol{a}, \boldsymbol{h}, N$ such that
  - $\mathcal{E}_{gen}$ is a linear combination of series of the form $\Theta_{\boldsymbol{a},\boldsymbol{h},N} | S_{M,m}$
    - Here, "$|S_{M,m}$" is the sieving operator acting on power series as

$$\left(\sum_{n\geq 0} a_n q^n\right)\Bigg| S_{M,m} := \sum_{n \equiv m \ (\mathrm{mod}\ M)} a_n q^n \text{, where } m, M \in \mathbb{N}$$

  - U_{spn} consists of a single unary theta function.
- The computer searched for identities by trying to find patterns in the ratio $r_{\boldsymbol{a},\boldsymbol{h},N}(n)/r_{\boldsymbol{a}}(n)$ for n that were not in certain 'bad' square classes; if a consistent repeating pattern was found, the computer would flag these values of $\boldsymbol{a}, \boldsymbol{h}, N$.
- By the theory of modular forms, only a finite number of coefficients needed to be checked to prove the identity.

For Goal II: Spinor Classification

- All classes in the genus were found first, using the techniques of Sun from [5].
- These classes were then partitioned into different spinor genera; the technique involved here was essentially the same as that used by Haensch and Kane in [4], along with certain results of Xu [6], and Conway and Sloane [7].

Results

Number of $\boldsymbol{a}, \boldsymbol{h}, N$ flagged by computer programme	**202**
Number of these $\boldsymbol{a}, \boldsymbol{h}, N$ that satisfied an identity of the required type	**85**
Number of these $\boldsymbol{a}, \boldsymbol{h}, N$ whose corresponding genus of cosets were classified	**40**
Number of these $\boldsymbol{a}, \boldsymbol{h}, N$ whose corresponding shifted lattice was spinor class 1	**37**

An example of the kind of identity that was found is

$$\Theta_{\boldsymbol{a},\boldsymbol{h},4} = \frac{1}{12}\Theta_{\boldsymbol{a}}|S_{8,1} + \frac{1}{2}\theta_{\chi_{-4},1} \quad \text{where} \quad \boldsymbol{a} = (1,1,1)^T \text{ and } \boldsymbol{h} = (1,0,0)^T$$

This gives the following relationship between $r_{\boldsymbol{a},\boldsymbol{h},N}(n)$ and $r_{\boldsymbol{a}}(n)$:

$$r_{\boldsymbol{a},\boldsymbol{h},N}(n) = \begin{cases} 0 & \text{if 8 does not divide } n-1 \\ \frac{1}{12} r_{\boldsymbol{a}}(n) & \text{if } 8|(n-1) \text{ and } n \text{ is not a perfect square} \\ \frac{1}{12} r_{\boldsymbol{a}}(n) + \frac{1}{2}(-1)^{(k-1)/2}k & \text{if } 8|(n-1) \text{ and } n = k^2 \text{ for some } k \in \mathbb{N} \end{cases}$$

Some more examples of more complicated identities are:

$$\Theta_{(1,3,3)^T,(3,3,4)^T,12} = \Theta_{(1,3,3)^T}\left|\left(\frac{1}{40}S_{144,84} + \frac{1}{48}S_{288,12} + \frac{1}{64}S_{288,156}\right)\right. - \frac{1}{4}\theta_{\chi_{-12},12}$$

$$\Theta_{(1,3,9)^T,(1,0,0)^T,2} = \Theta_{(1,3,9)^T}\left|\left(\frac{1}{2}S_{8,1} + \frac{1}{4}S_{8,5}\right)\right. + \theta_{\chi_{-12},1}$$

As an example of a genus classification, we consider the genus of the lattice corresponding to $\boldsymbol{a} = (1,1,1)^T$, $\boldsymbol{h} = (1,0,0)^T$ and $N = 4$:

Genus

Spinor Genus	
Class	
$(1,0,0)^T$	$(0,1,0)^T$
$(0,0,1)^T$	$(3,0,0)^T$
$(0,3,0)^T$	$(0,0,3)^T$

Spinor Genus	
Class	
$(1,2,2)^T$	$(2,1,2)^T$
$(2,2,1)^T$	$(3,2,2)^T$
$(2,3,2)^T$	$(2,2,3)^T$

(here, the values of $\boldsymbol{a}$ and N are the same for each shifted lattice; only $\boldsymbol{h}$ differs)

From this, it is reasonable to expect $\Theta_{\boldsymbol{a},(1,2,2)^T,4}$ ($\boldsymbol{a} = (1,1,1)^T$) satisfies a similar identity as $\Theta_{\boldsymbol{a},(1,0,0)^T,4}$; indeed,

$$\Theta_{\boldsymbol{a},(1,2,2)^T,4} = \frac{1}{12}\Theta_{\boldsymbol{a}}|S_{8,1} - \frac{1}{2}\theta_{\chi_{-4},1}$$

References

[1] O. Timothy O'Meara. Introduction to Quadratic Forms. Springer-Verlag, New York, 1973.

[2] Wai Kiu Chan and Byeong-Kweon Oh. Representations of integral quadratic polynomials, 2012.

[3] Kathrin Bringmann and Ben Kane. Class numbers and representations by ternary quadratic forms with congruence conditions. 2020.

[4] Anna Haensch and Ben Kane. Almost universal ternary sums of polygonal numbers, 2016.

[5] Liang Sun. Class numbers of quadratic diophantine equations. Journal of Number Theory, 166:181–192, 2016.

[6] Fei Xu. Strong approximation for certain quadric fibrations with compact fibers, 2014.

[7] John H. Conway and Neil J. Sloane. Sphere Packings, Lattices and Groups. Springer-Science+Business Media, 3 edition, 1999.