Mechanics and Manipulation of Active Structures

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A4 ORF Mathematics

Abstract

In this research, we studied stability of helical equilibria of an isotropic Kirchhoff elastic rod with clamped ends and with drift in general curved space. We proved that three of five control parameters thoroughly determine helical stability. We then proved that every helix is stable only at finite length, and derived a scaling relationship which helps to compute and visualize the boundary between stable and unstable helices in both spherical space and hyperbolic space.

Problem Formulation

The configuration of an inextensible and unshearable rod of length L in general curved space is described as a function $r : [0, L] \to S$ where $S \in \{\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3\}$ and control function R from $[0, L]$ to some Lie groups. For example, in Euclidean space, it is mapped to $SO(3)$, which is the set of three-dimensional rotation matrices. Coordinate-free formulation of the problem gives us a uniform expression of the control problem in the following way:

$$\begin{aligned} \underset{q,u}{\text{minimize}} \quad & \int_0^L \frac{1}{2}\left(c_1 u_1^2 + c_2 u_2^2 + c_3 u_3^2\right)\, ds \\ \text{subject to } & q' = q\,\zeta_\epsilon(u),\ q(0) = I_{4\times 4},\ q(L) = q_f. \end{aligned} \tag{1}$$

Here, the functional is the elastic energy of the isotropic elastic rod, and $c > 0$ describes its torsional and bending stiffness. By assuming the rod to be uniform, we restrict c to be constant. Further, u stands for twisting strain and bending strain in helices. We impose clamped boundary conditions on the rod, and q is a coordinate-free formulation of helices, which reserves complete information in them. $\zeta_\epsilon(u)$ is a 4×4 matrix. Its leading principle submatrix of order 3 is $\hat{u}$ with the map $\hat{}: \mathbb{R}^3 \to so(3)$ satisfying $a \times b = \hat{a}b$ for all $a, b \in \mathbb{R}^3$. It's zero elsewhere except $\zeta_\epsilon(1, 4) = 1$ and $\zeta_\epsilon(4, 1) = \epsilon$. Different ϵ corresponds to different curved space. Particularly, $\epsilon = 1$ corresponds to curves on $\mathbb{S}^3$ and $q(t)$ is an element of the Lie group $SO(4)$, while $\epsilon = -1$ corresponds to curves on $\mathbb{H}^3$ and $q(t)$ is an element of the Lie group $SO(1, 3)$. Setting $\epsilon = 0$ recovers the case $\mathbb{R}^3$, with $q(t)$ being an element of the special Euclidean group $SE(3)$ [1][2]. Our project aims to study optimality of helices with constant control.

Methodology

If (p, u) is a local minimum of the problem, by applying Pontryagin Maximum Principle, there exist function μ such that

$$\begin{aligned} \mu_1' &= \mu_2 u_3 - \mu_3 u_2 & \mu_4' &= \mu_5 u_3 - \mu_6 u_2 \\ \mu_2' &= \mu_3 u_1 - \mu_1 u_3 + \mu_6 & \mu_5' &= \mu_6 u_1 - \mu_4 u_3 + \epsilon\mu_3 \\ \mu_3' &= \mu_1 u_2 - \mu_2 u_1 - \mu_5 & \mu_6' &= \mu_4 u_2 - \mu_5 u_1 - \epsilon\mu_2, \end{aligned} \tag{2}$$

the functions u and μ are related by $u_1 = \mu_1/c$, and $u_i = \mu_i$ for $i = 2, 3$.

On the other hand, by Jacobi Conjugate Point Test, consider differential equations related to F, G, and H as

$$M' = FM \qquad J' = GM + HJ, \tag{3}$$

where

$$F = \begin{bmatrix} 0 & c_{32}\mu_3 & c_{32}\mu_2 & 0 & 0 & 0 \\ c_{13}\mu_3 & 0 & c_{13}\mu_1 & 0 & 0 & 1 \\ c_{21}\mu_2 & c_{21}\mu_1 & 0 & 0 & -1 & 0 \\ 0 & -c_2^{-1}\mu_6 & c_3^{-1}\mu_5 & 0 & u_3 & -u_2 \\ c_1^{-1}\mu_6 & 0 & \epsilon - c_3^{-1}\mu_4 & -u_3 & 0 & u_1 \\ -c_1^{-1}\mu_5 & c_2^{-1}\mu_4 - \epsilon & 0 & u_2 & -u_1 & 0 \end{bmatrix}$$

$$G = \mathrm{diag}(c_1^{-1}, c_2^{-1}, c_3^{-1}, 0, 0, 0)$$

$$H = \begin{bmatrix} 0 & u_3 & -u_2 & 0 & 0 & 0 \\ -u_3 & 0 & u_1 & 0 & 0 & \epsilon \\ u_2 & -u_1 & 0 & 0 & -\epsilon & 0 \\ 0 & 0 & 0 & 0 & u_3 & -u_2 \\ 0 & 0 & 1 & -u_3 & 0 & u_1 \\ 0 & -1 & 0 & u_2 & -u_1 & 0 \end{bmatrix}.$$

where we have used notation $c_{ij} = c_i^{-1} - c_j^{-1}$.

If a solution to the Hamiltonian equations equipped with original problems and the above differential equations satisfies $\det(J(t)) \neq 0$ for all $t \in [0, L]$, then that solution is a local minimum of the optimal control problem. Otherwise, it is not a local minimum of the optimal control problem.

In fact, two control parameters curvature κ and torsion τ are functions of μ:

$$\kappa = \sqrt{\mu_2^2 + \mu_3^2} \qquad \tau = \mu_1 - \frac{\mu_2\mu_5 + \mu_3\mu_6}{\mu_2^2 + \mu_3^2}.$$

We want to study optimal configuration of helices with constant control, and we ask problems like - how do curvature κ and torsion τ change the optimality of helices? Is there any other parameters that determine the configuration or optimality of helices?

Main Results

In our study we mainly focus on case of $c_2 = c_3 = 1$. We denote c_1 as c, which is the ratio of torsional to bending stiffness in this scenerio. In fact, other cases follow exactly the same idea.

(i). Expression of μ:

If $\kappa > 0$ and $\tau \in \mathbb{R}$, $a_1 \in \mathbb{R}$, and $\phi_0 \in [0, 2\pi)$ are given. Define

$$\begin{aligned} a_2 = \kappa cos(\phi_0), \quad a_3 = \kappa sin(\phi_0), \quad a_4 = \tau(a_1 - \tau) + \varepsilon, \\ a_5 = a_2(a_1 - \tau), \quad a_6 = a_3(a_1 - \tau) \end{aligned} \tag{4}$$

Then the solution of 2 with $\mu(0) = a$ is given by

$$\begin{aligned} \mu_1 &= a_1, & \mu_4 &= \tau(a_1 - \tau) + \varepsilon, \\ \mu_2 &= \kappa cos(\gamma t + \phi_0), & \mu_5 &= \mu_2(a_1 - \tau), \\ \mu_3 &= \kappa sin(\gamma t + \phi_0), & \mu_6 &= \mu_3(a_1 - \tau). \end{aligned} \tag{5}$$

with constant curvature κ and torsion τ.

For now on, we denote a_1 as ω, since this parameter is related to twisting of helices. Now it seems the stability is related to five parameters c,ϕ_0, ω, κ, τ.

(ii). Optimality is independent of ϕ_0:

To reduce ϕ_0, by coordinate transformation

$$\tilde{J}(s) = K(s)J(s) \quad \tilde{M}(s) = K(s)M(s) \tag{6}$$

where $K(s) = \begin{bmatrix} K_0 & O_{3\times 3} \\ O_{3\times 3} & K_0 \end{bmatrix}$, $K_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta(s)) & sin(\theta(s)) \\ 0 & -sin(\theta(s)) & cos(\theta(s)) \end{bmatrix}$.

and $\theta(s) = \gamma s + \phi_0$. Then,

$$\tilde{J}'(s) = \tilde{H}\tilde{J} + \tilde{G}\tilde{M}, \quad \tilde{M}'(s) = \tilde{F}\tilde{M}, \tag{7}$$

Since $det(K(s)) = 1$ for all $s \in [0, L]$, conjugate points are invariant under the coordinate transformation.[5] Also, $\tilde{H}$,$\tilde{F}$ and $\tilde{G}$ is independent of ϕ_0, we conclude that conjugate points are independent of the parameter ϕ_0. Then we can assume that $\phi_0 = 0$ without loss of generality.

(iii). Optimality is independent of c:

We observed that, if $\tilde{J}(0) = 0_{6\times 6}$ and $\tilde{M}(0) = I_{6\times 6}$. If $\tilde{M}(0)$ is changed from $I_{6\times 6}$ to R, then the resulting solution is given by $\tilde{J}R$ and $\tilde{M}R$. Therefore, if we pick a different initial condition $\tilde{M}(0)$, the conjugate point test of $\tilde{J}$ will not be affected. We proved that, for every choice of initial condition $\tilde{M}(0)$, only the first column of $\tilde{J}(t)$ is dependent on c, therefore, we want to pick an invertible initial condition $\tilde{M}(0)$ such that, the first column of of $\tilde{J}(t)$ is of the form $\begin{bmatrix} g(c) \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix}^t$, where $g(c)$ is some function of c. By solving odes, we find that the first column of $\tilde{M}(t)$ equals to $\begin{bmatrix} 1 \ 0 \ -\frac{\kappa}{c}t \ \tau \ \kappa \ (\tau - \omega)\frac{\kappa}{c}t \end{bmatrix}^t$, therefore we need to pick $\tilde{M}(0) = I_{6\times 6} + \begin{bmatrix} 0 \ 0 \ 0 \ \tau \ \kappa \ 0 \end{bmatrix}' \times \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix}$

Then $\tilde{J} = \begin{bmatrix} \frac{t}{c} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix}$ where $\tilde{J}_{22}$ is a 5×5 matrix without c. $det(\tilde{J}(t)) = \frac{t}{c}det(\tilde{J}_{22}(t))$. We can see from this that c does not affect the conjugate point test.

(iv). Finite stable length:

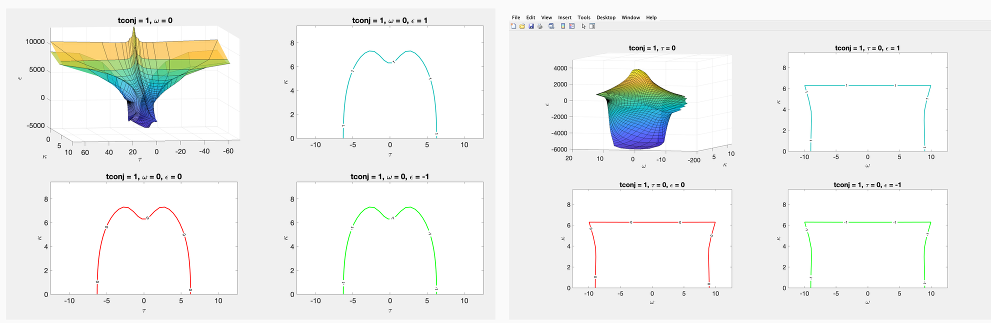
Now for every constant control κ, τ and ω, we want to prove that the helices become unstable at finite length, or equivalently, we have finite minimum conjugate time. For Jordan Normal form of $A = \begin{bmatrix} \tilde{F} & O_{6\times 6} \\ \tilde{G} & \tilde{H} \end{bmatrix}$, there are two Jordan blocks with the form $\pm(-(a_1 - 2\tau)^2 - \kappa^2)$, which are odd dimensional Jordan blocks with pure imaginary eigenvalue, applying results in [3], the minimal conjugate time is always finite.

(v). Scaling property:

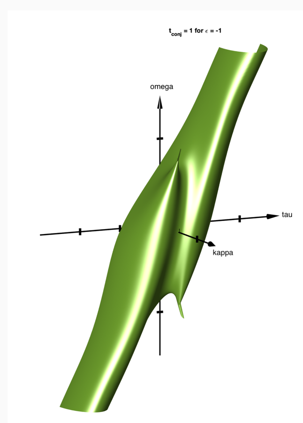
It is really difficult to determine each minimum conjugate time for every constant control. Instead, we try to find some scaling property to help us simplify the calculation. Given $\kappa \in \mathbb{R}^+$ and ϵ, τ, $\omega \in \mathbb{R}$, let $S_c(\kappa, \tau, \omega, \epsilon)$ denote the first conjugate time. We have:

$$S_c(\lambda\kappa, \lambda\tau, \lambda\omega, \lambda^2\epsilon) = \frac{1}{\lambda}S_c(\kappa, \tau, \omega, \epsilon)$$

Particularly, this means that for $\epsilon = 0$, every array through the origin will intersect the boundary at most once. We set the final length $L = 1$. Then in $\kappa - \tau - \omega$ space, for every arrow starting from zero pointing to positive κ axis, it hits the boundary between the optimal and non-optimal shape exactly once. However, when $\epsilon \neq 0$, there are four parameters changing in the scaling property, including ϵ, and is difficult to visualize the results. Also, in these cases, each arrow may hits the boundary more than once,[4] and we want to find the smallest one. We can set $\omega = 0$ or $\tau = 0$ respectively, to see how does the minimum conjugate time behave at $\epsilon = \pm 1$. Results are as follow:



Our final step is trying to plot the surface of $tconj = 1$ in the case of $\epsilon = \pm 1$ (That is, find the boundary point between optimal and non-optimal with final length equal to 1). It turns out, the difference of surface for $tconj = 1$ between $\epsilon = 0$ and $\epsilon = \pm 1$ is extremely small - for each array, the scaling is between 0.95 to 1.05, as we can see, all these surfaces look similar. They are of the following shape:



Conclusion

In our project, we studied the optimality of helices. In 3D, we have shown that the stiffness parameter c does not affect the conjugate point test, and every helices with constant control have finite minimum conjugate time. We found that the surface for $tconj = 1$ is similar in all three different curved spaces. In our future work, we may study other geometric optimal control problem with $\epsilon \neq 0$ or ± 1. Acknowledgement: Finally, I would like to say thanks to Dr. Andy Borum who gives us a lot of help and guidance throughout this summer.

References

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