New Light on the Mysteries of Primes

by Dr LAU Yuk Kam, Department of Mathematics

If you are a movie buff, especially on mystery dramas, you may have watched the two films “Perfect Number” and “Suspect X” adapted from the same novel “The Devotion of Suspect X” (嫌疑犯的獻身) of the popular series “Detective Galileo” (神探伽利略). The story is about a math teacher who ever has a dream of solving the Goldbach conjecture or the Riemann hypothesis. These two great problems relate intimately to the fundamental elements of natural numbers, called primes, such as 2, 3, 5, 7, 11, 13, 17 ....... (Recall that P is a prime if P is a natural number that has exactly two positive divisors 1 and P.)

As well, you might recall a plot of the movie “The Solitude of Prime Numbers” (質數的孤寂), saying “The primes are suspicious, solitary numbers. Perhaps they would prefer to be ordinary numbers, but for some reason they could not do it. There are some that almost touch — 17 and 19, 41 and 43 — but are separated by an even number. As the numbers expand, these so-called twin primes become ever rarer, and the presentiment develops that these were accidents, and that solitude is the true destiny. But then, if you count long enough, you’ll find another pair of twins, clutching each other tightly.” (The speculated endless occurrence of twin primes is the Twin Prime Conjecture.)

In the Second International Congress of Mathematicians (of the year 1900), David Hilbert posed a list of 23 contemporary unsolved problems and the 8th one comprised three big open problems in number theory: Riemann Hypothesis, the Goldbach conjecture and the Twin Prime conjecture.

The Goldbach Conjecture is one of the puzzles attracting numerous amateur mathematicians, perhaps because it is easy to understand. The conjecture is originated from a letter of Christian Goldbach to Leonhard Euler in the year 1742 and actually consists of two parts:

**G(2).**

**Strong Goldbach Conjecture**

Every even integer $n > 2$ is a sum of two primes $p$ and $p'$, i.e.

$$n = p + p'$$

For example,

$$8 = 3 + 5$$

and

$$16 = 5 + 11.$$  

**G(3).**

**Weak Goldbach Conjecture**

Every odd integer $n > 5$ is a sum of three primes $p$, $p'$ and $p''$, i.e.

$$n = p + p' + p''$$

For example,

$$7 = 2 + 2 + 3$$

and

$$19 = 3 + 5 + 11.$$  

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*The film is based on the novel of the same name and was nominated for the Golden Lion at the 67th Venice International Film Festival. The author Paolo Giordano, who was born in 1982 and got a doctoral degree in theoretical particle physics, won the Strega Prize — the most prestigious Italian literary award — in 2008 with this novel.*
Two consecutive primes $P, P_{next}$ form (a pair of) twin primes if the length of this prime gap, i.e. $P_{next} - P$, equals 2. For instance, 3 and 5, 11 and 13, 431 and 433, 1997 and 1999 are twin primes. So the twin prime conjecture, saying that there are infinitely many pairs of twin primes, can be rephrased as: prime gaps of length 2 appear unceasingly.

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The method of GPY makes use of the sieve theory which can be viewed as a mathematical fishing technique with various nets. Such a technique appears more than 2,000 years ago in Greek. The modern sieve methods are somewhat sophisticated. GPY developed a sieve which has a fundamental intimacy with the sieve method — an early approach to the twin prime and the Goldbach problem — invented by Atle Selberg in 1940s.

Selberg is a recipient of the Fields medal, commonly regarded as the Nobel Prize of mathematics. In 1998 he gave, as the Y.C. Wong Visiting Lecturer, a series of lectures on the analytic theory of the prime numbers at HKU.

Thus mathematicians study the size of small prime gaps. Not much is known in the 20th century. (More precisely, mathematicians look for a small threshold so that infinitely many prime gaps, $P_{next} - P$, do not exceed the threshold amount. Nobody could suppress the threshold below 0.24 (log $P$) in the 20th century.)

In 2003, Daniel Goldston and Cem Yildirim announced a landmark result but shortly afterwards a fatal error was found. The salvage was finally succeeded in 2005 with the help of János Pintz, and Goldston-Pintz-Yildirim (GPY) obtained the following:

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The theorem of GPY in 2005 gives infinitely many small prime gaps; however the size of these gaps, $p_{n+1} - p_n$, may grow to infinity. Qualitatively it is very far from the twin prime conjecture (which predicts infinitely many prime gaps with $p_{n+1} - p_n = 2$).

On the other hand, GPY pointed out in their paper, “[our theorem] would appear to be within a hair’s breath of obtaining [bounded prime gaps].” [Bounded prime gaps mean that all these gaps have lengths not exceeding a constant threshold, i.e. all such $p_{n+1} - p_n$ are less than a constant value.] This remark impressed Yitang Zhang (張益唐) deeply and Zhang started thinking of the problem.

After working solely for several years, Zhang overcome the barrier underlying in the work of GPY and accomplished in 2013 the groundbreaking result:

Although it does not guarantee infinitely many prime twins (which require the gap length of 2), Zhang’s theorem assures infinitely many “prime cousins”.

Zhang’s work was widely spread in May of 2013 and drew many mathematicians’ attention, not only because of its beauty and significance but also how much the bound size 70,000,000 can be reduced. “This result is, of course, not optimal”, Zhang wrote in his paper.

On June 4 of 2013, Terence Tao (陶哲軒), a Fields medalist, launched the Polymath8 project to improve Zhang’s bound. Polymath project is an open online platform for massive collaboration. After a few months, the first phase of the project improved the number 70,000,000 to 4680 by developing the techniques of Zhang. The second phase incorporated the work of a young researcher, James Maynard, which gives a novel and efficient sieve method, to further improve the size to 270.

Amazingly the seed of Maynard’s idea also lies in the paper of Selberg on his sieve theory. Will the size be lowered to 2 to prove the twin prime conjecture? Maynard said, “I feel that we still need some very large conceptual breakthrough to handle the twin primes case.”

Nevertheless, such a breakthrough may sneak out as quietly and suddenly as Zhang’s!

### Epilogue

Primes have many mysteries, some of which lead to modern applications. As well known, public-key cryptography is a vital technique for Internet security and it makes use of number theory and large primes. The digital rights group “EFF” offers a prize of US$250,000 to the first individual or group who discovers a prime with one billion (or more) decimal digits.

However mathematicians are fascinated by the pattern of primes for simple reasons. “The problem is what attracts us, Pure Mathematicians. If a problem is useful, [it is] good; if not useful, [it is] also good as long as it is fundamental and deep,” said Peter Sarnak, Professor of the Institute for Advanced Study in Princeton, Chairman of Selection Committee for the Shaw Prize (邵逸夫獎) in Mathematical Sciences.

### Brainteaser

**Question of last issue**

Coins have many uses apart from being a medium of payment to facilitate transactions. For instance, coin tossing is a simple and quick way of settling disputes in a civilized way or deciding between two options. To serve such purposes, the coin used has to be fair so that there is an equal chance of getting a head or a tail. Suppose a coin is given but we are not sure whether it is fair. What can we do? How can a possibly unfair coin be used “in a fair way”? You may assume that the tosser is “fair” in tossing the coin each time so that there is no correlation between the outcomes.

**Answer**

If you toss the coin twice, you know that whatever the bias, HT (heads then tails) is as likely as TH. So you toss it twice. If it comes up HT, you select option A. If it comes up TH, you select option B. If it comes up HH or TT, you toss the coin twice again until you get either HT or TH. Using this procedure, the chance of selecting option A or B will be the same. Try yourself by repeating the procedures many times, and you will obtain roughly the same number of A and B.

### The 10 Largest Known Primes

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The successful soft landing of China’s Chang’e 3 mission on our nearest celestial neighbour, the Moon, on December 14, 2013 – the first one since 1976 – represents a major milestone in lunar exploration. The Chinese space programme aims to return robotically samples of rocks and soil from the Moon as early as 2017, paving the way for astronauts to set foot on the Moon again in the future. As with the space race between the United States and the Soviet Union that eventually led to the Apollo astronauts’ visits to the Moon between 1969 and 1972, there is a political dimension to China’s space programme. But it is important to note that human lunar exploration have had, and will continue to have, major impacts on our understanding of the formation and evolution of not just the Moon, but of the entire Solar System.

One of the scientific legacies of the manned Apollo and robotic Soviet missions are the lunar samples returned to Earth. Laboratory analysis of the samples showed that the composition of the Moon is different from that of the Earth. It is depleted in volatile elements that have relatively low condensation temperature, as well as siderophile elements that tend to affiliate with iron, but it is enriched in refractory elements that have relatively high condensation temperature. To explain this unusual composition of the Moon, the masses of the Earth and Moon, and the amount of rotation (angular momentum) in the Earth-Moon system, the giant impact theory, where the Moon accreted in a debris disk generated by the giant impact of a Mars-sized planetary embryo with the proto-Earth about 4.5 billion years ago, was proposed and is now widely accepted. Furthermore, large impacts and frequent impacts are now considered to be a fundamental process in planet formation.

Impacts on both the Earth and the Moon have continued to the present time, as recorded by the numerous impact craters on the Moon. Radiometric dating of samples obtained from different areas of the Moon allows scientist to study how the rate of impacts has changed with time. The impact rate was much higher in the early Solar System due to the presence of a lot of debris left over from planet formation. Surprisingly, some data suggest that there was a peak in the impact rate about 3.9 billion years ago or about 700 million years after the planets formed. This so-called Late Heavy Bombardment is one of the puzzling properties of our Solar System that led to the radical idea that the orbits of the giant planets in the outer Solar System were originally much closer together. According to the Nice model (named after the city Nice in France) proposed in 2005, the slow migration of the orbits of Jupiter and Saturn eventually led to a sudden large scale instability in the outer Solar System, in which Uranus and Neptune were scattered to orbits much further from the Sun. The Late Heavy Bombardment could be explained by this global instability scattering many asteroids and comets into the inner Solar System. Recent models in which the outer Solar System started with five giant planets and lost one during the instability may work even better than the original Nice model.

Despite these major advances in our understanding of the Moon and the Solar System, key questions remain. Recent re-analysis of lunar samples using modern laboratory techniques has shown that the Moon may have more water than previously thought. The evidence for the Late Heavy Bombardment on the Moon also remains controversial. The collection of samples from new areas of the Moon by future missions may help scientists to answer these questions and lead to new puzzles.
Some materials have exotic conducting edge or surfaces while their inner part are insulating, looking like good insulators covered by a thin layer of good metal (see Figure). The conducting boundaries are formed by the topological effects, which are insensitive to the shape or geometry of materials. Even when a material is cut into two halves, the new interfaces are still conducting. Such novel materials are named topological insulators, which provide new routes to generate new quantum phases or particles. Exotic electric and magnetic properties of these materials will have potential applications in the next generation of electronic devices and topological quantum computation.

Studies of topological matters were dated back to 1970s and 80s. Discovery of the quantum Hall effect opened a new window to explore novel quantum states of matter. Almost at the same period it was found that the conducting polymers have very special structures, and their charge carriers are the domain walls of two distinct one-dimensional structures, and reveal topological properties. In 1972 it was observed that liquid helium He3 becomes superfluiding, and two different phases at very low temperatures. The three irrelevant topics more than thirty years ago are now realized to have the same topological origins, and belong to the same family of topological matters.

A milestone in the field is the prediction of the quantum spin Hall effect by Kane and Mele in 2005. The effect was soon verified experimentally and generalized to three dimensions. A series of materials are discovered to be topological insulators, which initiates a “Gold Rush” in the search of topological matters. Professor Shunqing Shen and his group entered this field at the early stage, and made significant contributions. In 2004 he proposed a relation between the Berry phase and quantum spin transport. In 2005 he discovered a new type of force on moving electron spin in an electric field, which plays an essential role in quantum spin transport. In 2008, he and his collaborators found the finite size effect of the boundary states, which was soon verified experimentally. In 2009, he and his students discovered a new type of topological insulators driven by disorders or impurities, named topological Anderson insulator, which becomes now a topic in the field. In 2011 he and his collaborators proposed a quantum transport theory for topological insulators and predicted a crossover from weak anti-localization to weak localization, which also has been observed experimentally by several groups.

In 2010, Professor Shen realized a single mathematical equation as a key to understand the topological matters. The equation is a modification to the famous Dirac equation for relativistic quantum mechanics. He found that a large class of topological matters could be well described by this equation. It covers materials from one, two, to three dimensions, and from insulators to superconductors, or superfluids. The main results are summarized in his single-authored monograph, Topological Insulators (Springer Series in Solid State Science 174, 2012), which is the first such publication on the frontier topic.
Dr Moriaki Yasuhara (School of Biological Sciences, Swire Institute of Marine Science, and Department of Earth Sciences), in collaboration with Smithsonian and US Geological Survey researchers, discovered that dynamic changes in large-scale geographic patterns of number of species (biodiversity) are driven by climate, especially temperature, and there has been a strong and remarkable stability in the relationship between biodiversity and temperature over the last three million years. Their finding has important implications for global warming as the very strong temperature control of biodiversity suggests that future temperature rise will alter present-day ecosystems. Thus, scientists can now predict that this warming will give rise to substantial local changes in biodiversity. The result was published in *Ecology Letters* in October 2012.

Planet Earth shows striking latitudinal gradients in species richness and biodiversity with, typically, more species in tropical regions and fewer in polar regions. For instance, tropical rainforest contains many more species than forests further north, and coral reefs host many more species than Arctic Ocean. Understanding the causes, stability and potential changes through time of such latitudinal species gradients is limited. By studying fossil of planktic marine animals from more than 200 sites covering the whole North Atlantic Ocean, and using it as a model system to reveal global patterns, the research team discovered that the quantitative relationship between diversity and latitude has been dynamic, but the diversity versus temperature relationship has remained remarkably constant throughout the past three million years. Dr Yasuhara and his research team compared three time periods of present day, last ice age (colder than today), and Pliocene (3 million years ago: warmer than today), and temperature differences among these periods are clearly reflected in diversity. In other words, warmer Pliocene had higher diversity and colder last ice age has lower diversity than the present day. Furthermore, the latitudinal temperature gradient was steeper during the last ice age than today because of polar regions were much cooler in the past than they are now and tropical regions were not much, and thus the latitudinal species diversity gradient was steeper. So, all of such changes over time are reasonably explained by the underlying constant biodiversity-temperature relationship (see Figure). These results provide strong evidence that temperature controls biodiversity.

“Our results will largely advance our understanding on mechanisms controlling biodiversity. This is a very important topic in our rapidly changing world with serious concern on biodiversity conservation”, said Dr Yasuhara. It remains an open question as to whether these results will allow scientists to predict biodiversity changes in the future, since the rates of global warming and temperature rises are beyond those seen in the last three million years. However, the findings certainly indicate that biodiversity will change in response to temperature, and these temperatures beyond the historic ranges may well result in substantial reorganization of global pattern of biodiversity.

This figure is a simplified graph to explain latitude-biodiversity (left) and temperature-biodiversity (right) relationships. Note large differences of the latitudinal diversity patterns among the three time slices of present day (yellow), last ice age (18,000 years ago, a period colder than today: blue), and 3 million years ago (Pliocene, a period warmer than today: red), and remarkable similarity of the temperature-diversity patterns. Temperature-diversity lines of three time slices are well overlapped, suggesting strong and constant temperature control of biodiversity.
“Science without Borders” is a science service trip organized by the HKU Science Outreach Team. A total of 16 students from diverse science disciplines went to Heyuan City, Guangdong province from Dec 25 to 28, 2013, spending their Christmas with Mainland pupils from rural area. One of the aims of this project is to encourage students from the Faculty to make use of their science knowledge to promote science to general public. At the same time, we also strive to inspire the underprivileged kids with simple experiments in order to arouse their interests towards science. The project is supported by the Faculty of Science and is funded by SERVICE 100 Fund.

Eight workshops were offered to over 200 students of Heyuan Yangming Primary School. Each workshop featured mini-experiments in 4 major themes: Microscope, Optics, Pressure and Magnetism. Kids were all thrilled to observe magnified hair and leaves under microscopes. Some of them were amazed by the optics illusion created by simple apparatuses such as lenses, mirrors and beakers of water. Others kept playing around with the electromagnets and railgun. Not to mention the exciting suction cup battles, they all loved what we did for them.

After the first day workshop, the teachers reflected that their students enjoy the experiments very much and requested for an extra session after school for the preschool kids on the second day. Considering the age of the kids, instead of teaching science, we tried to impress them. They were very happy seeing beautiful light rays generated by laser pointers, ray boxes and prisms, etc. We could see eyes full of curiosity. Kids were just inspired by the beauty of nature, while we were inspired by their instinctive curiosity and their passion to ask why. All our effort seemed to pay off when we were surrounded with so many happy faces.

What we have done in the trip was like igniting a fire with sparkles, and we should think about how to keep up the flame. Witnessing the contrast between the kids’ enthusiasm for science and their compromising situation of science education, we were eager to help improve the status quo with our consecutive contributions.

Regular service trips and stable connection with rural schools are important in enlightening the kids with science. Furthermore, we should even consider extending our attention to other underprivileged children in Asia. Where are we going? No one knows for sure. We just want to keep inspiring others and getting inspired.

You are welcome to revisit the highlights of our service trip at:
http://www.youtube.com/watch?v=cFrG01UdyXY

or find us on facebook:
https://www.facebook.com/scienceoutreach.hku
Department of Physics, received the Faculty of Science Award for Teaching Excellence 2012-13, for her outstanding teaching performance and the continuous efforts she has put in arousing students' learning interests.

Dr Philip YU, Department of Statistics and Actuarial Science, received the Faculty of Science Award for Service Contribution 2012-13, for his contributions in service, departmental administration, and outreach activities in the Faculty.

Miss CHAN Bin San Vera, PhD candidate supervised by Dr Vengatesan Thiagarajan of School of Biological Sciences, received the Faculty of Science Excellent Teaching Assistant Award 2012-13, for her contributions in outstanding performance in providing teaching support and interaction with students.

Dr Jason PUN Chun Shing, Department of Physics, received the Faculty Knowledge Exchange (KE) Award for his outreach project on light pollution entitled “Dimming the “Bright Pearl – Informing the Public on Light Pollution”.

Dr Philip YU, Department of Statistics and Actuarial Science, received the Faculty of Science Award for Service Contribution 2012-13, for his contributions in service, departmental administration, and outreach activities in the Faculty.

Others

Professor Zidan WANG, Department of Physics, shared the 2013 China National Award in Natural Sciences (2nd class) with Siliang ZHU of South China Normal University.

Professor Sun KWOK, Dean of Science, received the Outstanding Achievement Award from his alma mater, the University of Minnesota. This award recognizes graduates who have attained unusual distinction in their chosen fields or professions, or in public service, and who have demonstrated outstanding achievement and leadership on a community, state, national, or international level.

Professor CHUNG Lung Sang, of Department of Earth Sciences, was selected by the Local and International Editorial Boards of the Hong Kong Medical Journal as the Winner of the Best Original Paper Award this year for his article entitled “Utility of infrared thermography for screening febrile subjects”.

Professor Frederick LEUNG Koon Shing, Mathematics alumnus and Professor of Faculty of Education, was awarded the Hans Freudenthal Medal for 2013 by the International Commission on Mathematical Instruction, in recognition of his distinguished research in the areas of comparative studies of mathematics education.

Mr Kaineen CHEN, a high school student who has worked with Dr Kono LEMKE in the Department of Earth Sciences on a geochemistry research project for more than a year, attended AGU’s Fall Meeting, the largest worldwide conference in the geophysical sciences in San Francisco in December, 2013. Apart from attending talks and poster sessions, Mr Chen presented his own poster entitled “An ESI/FTICR- MS Study of Zinc Sulfate” as part of AGU’s Bright Students Training as Research Scientists (Bright STaRS) programme.